We were discussing multilinear maps.

Example 1: (dot product)

$$\mathbb{R}^n \times \mathbb{R}^n, \quad v, w \in \mathbb{R}^n$$

Define a bilinear map on $\mathbb{R}^n$ by

$$(\cdot, \cdot) : \mathbb{R}^n \to \mathbb{R}$$

$$(v, w) = \sum_{i=1}^{n} v_i w_i = v \cdot w$$

if $v = (v_1, \ldots, v_n)$, $w = (w_1, \ldots, w_n)$

The difference between this and example 1 last time is that v and w are both allowed to vary here.

Check bilinearity!

let $u = (u_1, \dots, u_n)$.

$$(u+v, w) = \sum_{i=1}^{n} (u_i + v_i) w_i$$

$$= \sum_{i=1}^{n} u_i w_i + \sum_{i=1}^{n} v_i w_i$$

$$= (u, w) + (v, w)$$

Linearity in the second coordinate is a similar calculation. Now if $c \in \mathbb{R}$,

$$(cv, w) = \sum_{i=1}^{n} (cv_i) w_i$$

$$= c \sum_{i=1}^{n} v_i w_i$$

$$= c(v, w),$$

second coordinate is similar.

Observe: $(\cdot,\cdot)$ is not linear as a map from $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

For example, if $(e_i)_{i=1}^n$ is the standard basis of $\mathbb{R}^n$,

$$(e_1, 0) + (0, e_1) = 0$$

but $(e_1, e_1) = 1$.

Example 2: (determinants)

Let $A = (a_{i,j})_{i,j=1}^{n}$

be an $n \times n$ matrix.

Let a_i, $1 \le i \le n$, be the vector

$$a_i = (a_{1,i}, a_{2,i}, \dots, a_{n,i})$$

(columns of A)

Define $\det(A)$ as a multilinear map from

$$\underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n \text{ times}}$$

$$\det(a_1, a_2, \ldots, a_n)$$

$$= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$

$$\in \mathbb{R}$$

Where

$S_n =$ all bijections from $\{1, 2, \cdots, n\}$ to itself.

Each such element can be expressed as a product of transpositions. A transposition is a map from $\{1, 2, \ldots, n\}$ to itself that fixes all but two elements and interchanges the remaining 2.

Example: $n=4$

$$\{1,2,3,4\} \xrightarrow{\sigma} \{1,2,3,4\}$$

$$\sigma(1)=1$$
$$\sigma(2)=4$$
$$\sigma(3)=3$$
$$\sigma(4)=2$$

is a transposition.

Although any $\sigma \in S_n$ is a product of transpositions, the representation is not unique (unless $n=1$).

However, the parity (even or odd number of transpositions) is unique.

Define

$$\text{Sign}(\sigma) = \begin{cases} 1, & \sigma \text{ is a product of an even number of transpositions} \\ -1, & \sigma \text{ a product of an odd number} \end{cases}$$

Observe, then, that if σ is a transposition, $\operatorname{sign}(\sigma) = -1$.

Moreover, if $\sigma, \tau \in S_n$,

$$\boxed{\operatorname{sign}(\sigma \circ \tau) = \operatorname{sign}(\sigma)\operatorname{sign}(\tau)}$$

Let's show det is multilinear. Look at the first coordinate.

$$\det(a_1 + b_1, a_2, a_3, \ldots, a_n)$$

$$= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \left(a_{1,\sigma(1)} + b_{1,\sigma(1)}\right) \cdot a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$

$$= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \left(a_{1,\sigma(1)} \cdot a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}\right)$$
$$+ \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \left(b_{1,\sigma(1)} \cdot a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}\right)$$

$$= \det(a_1, a_2, \cdots, a_n) + \det(b_1, b_2, \cdots, b_n)$$

A similar calculation holds for the other entries, scalar multiplication follows from the definition:

Let $c \in \mathbb{R}$.

$$\det(ca_1, a_2, \dots, a_n)$$

$$= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, c a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$

$$= c \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$$

$$= c \det(a_1, a_2, \dots, a_n).$$

Other coordinates similar.

Observe: If $\gamma \in S_n$,

then

$$\det(a_{\gamma(1)}, a_{\gamma(2)}, \ldots, a_{\gamma(n)})$$

$$= \text{sign}(\gamma) \det(a_1, a_2, \ldots, a_n)$$

In particular, if γ is a transposition, then

$$\det(a_{\gamma(1)}, a_{\gamma(2)}, \ldots, a_{\gamma(n)})$$

$$= -\det(a_1, a_2, \ldots, a_n)$$

$$\det(a_{\gamma(1)}, a_{\gamma(2)}, \ldots, a_{\gamma(n)})$$

$$= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, a_{1,\sigma\circ\gamma(1)}\, a_{2,\sigma\circ\gamma(2)} \cdots a_{n,\sigma\circ\gamma(n)}$$

$$= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\operatorname{sign}(\gamma)\operatorname{sign}(\gamma)\, a_{1,\sigma\circ\gamma(1)} \cdots a_{n,\sigma\circ\gamma(n)}$$

$$= \operatorname{sign}(\gamma) \sum_{\sigma \in S_n} \operatorname{sign}(\sigma\circ\gamma)\, a_{1,\sigma\circ\gamma(1)} \cdots a_{n,\sigma\circ\gamma(n)}$$

$$= \operatorname{sign}(\gamma) \sum_{\sigma\circ\gamma \in S_n} \operatorname{sign}(\sigma\circ\gamma)\, a_{1,\sigma\circ\gamma(1)} \cdots a_{n,\sigma\circ\gamma(n)}$$

(if we set $\tau = \sigma\circ\gamma$)

$$= \operatorname{sign}(\gamma) \sum_{\tau \in S_n} \operatorname{sign}(\tau)\, a_{1,\tau(1)} \cdots a_{n,\tau(n)}$$

$$= \operatorname{sign}(\gamma) \det(a_1, \ldots, a_n)$$

Such a multilinear form T, where

$$T(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = \operatorname{sign}(\sigma)\, T(v_1, \ldots, v_n)$$

is called alternating.